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Determination of m_s and $|V_{us}|$ from hadronic τ decays

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Abstract: The mass of the strange quark is determined from SU(3)-breaking effects in the τ hadronic width. Compared to previous analyses, the contributions from scalar and pseudoscalar spectral functions, which suffer from large perturbative corrections, are replaced by phenomenological parametrisations. This leads to a sizeable reduction of the uncertainties in the strange mass from τ decays. Nevertheless, the resulting m_s value is still rather sensitive to the moment of the invariant mass distribution which is used for the determination, as well as the size of the quark-mixing matrix element $|V_{us}|$. Imposing the unitarity fit for the CKM matrix, we obtain $m_s(2\,\mathrm{GeV}) = 117 \pm 17\,\mathrm{MeV}$, whereas for the present Particle Data Group average for $|V_{us}|$, we find $m_s(2\,\mathrm{GeV}) = 103 \pm 17\,\mathrm{MeV}$. On the other hand, using an average of m_s from other sources as an input, we are able to calculate the quark-mixing matrix element $|V_{us}|$, and we demonstrate that if the present measurement of the hadronic decay of the τ into strange particles is improved by a factor of two, the determination of $|V_{us}|$ is more precise than the current world average.

Keywords: Quark Masses and SM Parameters, QCD, Sum Rules.

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Contents

1. Introduction

About a decade after the development of QCD sum rules by Shifman, Vainshtein and Zakharov [1], it was realised that the hadronic decay of the τ lepton could serve as an ideal system to study low-energy QCD under rather clean conditions [2, 3, 4, 5]. Shortly afterwards, the experimental precision of τ decay data was sufficiently improved in order to explore this possibility in practice. Since then, detailed investigations of the τ hadronic width as well as invariant mass distributions have served to determine the QCD coupling α_s to a precision competitive with the current world average [6, 7, 8, 9]. More recently, the experimental separation of the Cabibbo-allowed decays and Cabibbo-suppressed modes into strange particles opened a means to also determine the mass of the strange quark [10]–[17], one of the fundamental QCD parameters within the Standard Model.

Until today, these strange mass determinations suffer from sizeable uncertainties due to higher order perturbative corrections. In the sum rule under investigation, scalar and pseudoscalar correlation functions contribute, which are known to be inflicted with large higher order QCD corrections [5, 18, 19, 10], and these corrections are additionally amplified by the particular weight functions which appear in the τ sum rule. As a natural continuation, it was realised that one remedy of the problem would be to replace the QCD expressions of scalar and pseudoscalar correlators by corresponding phenomenological hadronic parametrisations [12, 13, 15, 20], which are expected to be more precise than their QCD counterparts. In this work, we shall present a complete analysis of this approach, and it will be shown that the determination of the strange quark mass can indeed be significantly improved.

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By far the dominant contributions to the pseudoscalar correlators come from the kaon and the pion, which are very well known. The corresponding parameters for the next two higher excited states have been recently estimated [21]. Though much less precise, the corresponding contributions to the τ sum rule are suppressed, and thus under good theoretical control. The remaining strangeness-changing scalar spectral function has been extracted very recently from a study of S-wave $K\pi$ scattering [22, 23] in the framework of chiral perturbation theory (χPT) [24, 25] with explicit inclusion of resonances [26, 27]. The resulting scalar spectral function was then employed to directly determine the strange quark mass from a purely scalar QCD sum rule [28]. On the other hand, now we are also in a position to incorporate this contribution into the τ sum rule. The scalar ud spectral function is still only very poorly determined phenomenologically, but it is well suppressed by the small factor $(m_u - m_d)^2$ and can be safely neglected.

In the next section, for completeness, we begin by recalling the theoretical expressions which are required in the sum rules under investigation. Then, in section 3, the scalar and pseudoscalar contributions to the τ sum rules are investigated separately, and it is demonstrated that, although the uncertainties on the QCD side are large, they are well satisfied. In section 4, an improved determination of the mass of the strange quark is performed by replacing the QCD expressions for the scalar and pseudoscalar spectral functions with the corresponding hadronic parametrisations. Since the resulting strange quark mass depends quite sensitively on the value of the quark-mixing matrix element $|V_{us}|$, in section 5 we take a different approach to the considered sum rule.

Assuming a strange mass as extracted from other recent analyses, we use the τ sum rule to determine $|V_{us}|$. The resulting uncertainty on $|V_{us}|$ is still sizeable, but the error is dominated by the partial hadronic width of the τ decaying into strange particles. It is then demonstrated that an improvement of this measurement by a factor of two would result in a determination of $|V_{us}|$ better than the present world average, thus shedding some light on the question of the unitarity violation in the first row of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Finally, in section 6 we compare our results with other recent determinations of the strange quark mass and in section 7 we end with a discussion and conclusions.

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2. Theoretical framework

Below, we review the main theoretical expressions required in the theoretical analysis of the inclusive hadronic τ decay width. Further details and complete expressions can be found in the original works [5, 10, 13]. The central quantities in such an analysis are the two-point correlation functions

$$\Pi^{V/A}_{\mu\nu,ij}(p) \equiv i \int dx\, e^{ipx} \, \langle\Omega|\, T\left\{ J^{V/A}_{\mu,ij}(x)\, J^{V/A}_{\nu,ij}(0)^\dagger \right\} |\Omega\rangle\,, \tag{2.1}$$

where Ω denotes the physical vacuum and the hadronic vector/axialvector currents are given by $J^{V/A}_{\mu,ij}(x) = (\bar{q}_j \gamma_\mu (\gamma_5) q_i)(x)$. The indices i, j denote the light quark flavours up, down and strange. The correlators $\Pi^{V/A}_{\mu\nu,ij}(p)$ have the Lorentz decomposition

$$\Pi^{V/A}_{\mu\nu,ij}(p) = (p_\mu p_\nu - g_{\mu\nu}p^2)\,\Pi^{V/A,T}_{ij}(p^2) + p_\mu p_\nu\,\Pi^{V/A,L}_{ij}(p^2)\,, \tag{2.2}$$

where the superscripts in the transversal and longitudinal components denote the corresponding angular momentum $J = 1$ (T) and $J = 0$ (L) in the hadronic rest frame.

The hadronic decay rate of the τ lepton,

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \text{hadrons}\,\nu_\tau(\gamma))}{\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau(\gamma))} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}\,, \tag{2.3}$$

can be expressed as an integral of the spectral functions $\operatorname{Im}\Pi^T(s)$ and $\operatorname{Im}\Pi^L(s)$ over the invariant mass $s = p^2$ of the final state hadrons [29]:

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right)\operatorname{Im}\Pi^T(s) + \operatorname{Im}\Pi^L(s)\right]. \tag{2.4}$$

The appropriate combinations of the two-point correlation functions are given by

$$\Pi^J(s) \equiv |V_{ud}|^2\Big[\,\Pi^{V,J}_{ud}(s) + \Pi^{A,J}_{ud}(s)\,\Big] + |V_{us}|^2\Big[\Pi^{V,J}_{us}(s) + \Pi^{A,J}_{us}(s)\Big]\,, \tag{2.5}$$

with V_{ij} being the corresponding matrix elements of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix. As has been indicated in eq. (2.3), experimentally, one can disentangle vector from axialvector contributions in the Cabibbo-allowed ($\bar{u}d$) sector, whereas such a separation is problematic in the Cabibbo-suppressed ($\bar{u}s$) sector, since G-parity is not a good quantum number in modes with strange particles.

Additional information can be inferred from the measured invariant mass distribution of the final state hadrons. The corresponding moments R_τ^{kl}, defined by [30]

$$R_\tau^{kl} \equiv \int_0^{M_\tau^2} ds\left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds}\,, \tag{2.6}$$

can be calculated theoretically in analogy to $R_\tau = R_\tau^{00}$. Exploiting the analytic properties of $\Pi^J(s)$, the moments (2.6) can be expressed as contour integrals in the complex s-plane running counter clockwise around the circle $|s| = M_\tau^2$:

$$R_\tau^{kl} = -\,i\pi \oint_{|x|=1} \frac{dx}{x}\left[3\,\mathcal{F}^{kl}_{L+T}(x)\,\mathcal{D}^{L+T}(M_\tau^2 x) + 4\,\mathcal{F}^{kl}_{L}(x)\,\mathcal{D}^{L}(M_\tau^2 x)\right]. \tag{2.7}$$

Here, we have defined the new integration variable $x \equiv s/M_\tau^2$, and general expressions for the kinematical kernels $\mathcal{F}^{kl}_{L+T}(x)$ and $\mathcal{F}^{kl}_{L}(x)$ have been presented in ref. [13]. For the convenience of the reader, we have compiled explicit expressions for the moments $(0,0)$ to $(4,0)$, which will be utilised in our phenomenological analysis in table 1.

In the derivation of the above expression (2.7), integration by parts has been employed to rewrite R_τ^{kl} in terms of the physical correlators $\mathcal{D}^{L+T}(s)$ and $\mathcal{D}^L(s)$,

$$\mathcal{D}^{L+T}(s) \equiv -s\,\frac{d}{ds}\left[\Pi^{L+T}(s)\right], \qquad \mathcal{D}^{L}(s) \equiv \frac{s}{M_\tau^2}\,\frac{d}{ds}\left[s\,\Pi^{L}(s)\right], \tag{2.8}$$

(k,l)	$\mathcal{F}^{kl}_{L+T}(x)$	$\mathcal{F}^{kl}_{L}(x)$
(0,0)	$(1-x)^3\,(1+x)$	$(1-x)^3$
(1,0)	$\frac{1}{10}\,(1-x)^4\,(7+8x)$	$\frac{3}{4}\,(1-x)^4$
(2,0)	$\frac{2}{15}\,(1-x)^5\,(4+5x)$	$\frac{3}{5}\,(1-x)^5$
(3,0)	$\frac{1}{7}\,(1-x)^6\,(3+4x)$	$\frac{1}{2}\,(1-x)^6$
(4,0)	$\frac{1}{14}\,(1-x)^7\,(5+7x)$	$\frac{3}{7}\,(1-x)^7$

Table 1: Explicit expressions for the kinematical kernels $\mathcal{F}^{kl}_{L+T}(x)$ and $\mathcal{F}^{kl}_{L}(x)$ which will be used in our phenomenological analysis.

which both satisfy homogeneous renormalisation group equations, and thus eliminate the dependence on unphysical (renormalisation scale and scheme dependent) subtraction constants. For large enough negative s, the contributions to $\mathcal{D}^J(s)$ can be organised in the framework of the operator product expansion (OPE) in a series of local gauge-invariant operators of increasing dimension $D = 2n$ times appropriate inverse powers of s. This expansion is expected to be well behaved along the complex contour $|s| = M_\tau^2$, except at the crossing point with the positive real axis [31]. As can be seen from eq. (2.7) and table 1, however, the contribution near the physical cut at $s = M_\tau^2$ is strongly suppressed by a zero of order three or larger. Therefore, uncertainties associated with the use of the OPE near the time-like axis are expected to be very small.

Inserting the OPE series for $\mathcal{D}^J(s)$ into (2.7) and performing the contour integration, the resulting expression for R_τ^{kl} can be written as [5]

$$R_\tau^{kl} = 3\,(|V_{ud}|^2 + |V_{us}|^2) S_{\rm EW} \Big\{ 1 + \delta^{kl(0)} + \sum_{D\geq 2} \Big(\cos^2\theta_C\, \delta^{kl(D)}_{ud} + \sin^2\theta_C\, \delta^{kl(D)}_{us} \Big) \Big\} \qquad (2.9)$$

where $\sin^2\theta_C \equiv |V_{us}|^2/(|V_{ud}|^2 + |V_{us}|^2)$, θ_C being the Cabibbo angle, and the electroweak radiative correction $S_{\rm EW} = 1.0201 \pm 0.0003$ [32, 33, 34] has been pulled out explicitly. Owing to chiral symmetry, the purely perturbative dimension-zero contribution $\delta^{kl(0)}$ is identical for vector and axialvector parts. The symbols $\delta^{kl(D)}_{ij} \equiv (\delta^{kl(D)}_{ud,V} + \delta^{kl(D)}_{ud,A})/2$ stand for the average of vector and axialvector contributions from dimension $D \geq 2$ operators which contain implicit suppression factors $1/M_\tau^D$.

The separate measurement of Cabibbo-allowed and Cabibbo-suppressed decay widths of the τ lepton [12] allows one to pin down the flavour SU(3)-breaking effects, dominantly induced by the strange quark mass. Defining the difference

$$\delta R_\tau^{kl} \equiv \frac{R^{kl}_{\tau,V+A}}{|V_{ud}|^2} - \frac{R^{kl}_{\tau,S}}{|V_{us}|^2} = 3\, S_{\rm EW} \sum_{D\geq 2} \Big(\delta^{kl(D)}_{ud} - \delta^{kl(D)}_{us} \Big)\,, \qquad (2.10)$$

many theoretical uncertainties drop out since these observables vanish in the SU(3) limit. In particular, they are free of possible flavour-independent instanton as well as renormalon contributions which could mimic dimension-two corrections.

As was already stated in the introduction, the longitudinal contributions $\delta^{kl(2)}_{ij,L}$ are plagued with huge perturbative higher order corrections, which in previous analyses resulted in large corresponding uncertainties for the strange quark mass. Thus, in the following, we shall replace the theoretical expressions for the longitudinal part by corresponding phenomenological contributions, thereby strongly reducing those uncertainties in the m_s determination. Before we embark on the full investigation of the moment differences δR_τ^{kl}, to be able to judge the gain of the intended replacements in the longitudinal sector, in the next section we shall first investigate τ-like sum rules for the longitudinal part alone.

3. The longitudinal sector

From eq. (2.7), again making use of the analytic structure of the correlation functions, one can deduce the following τ-like sum rules for the longitudinal contributions:

$$R_{ij,V/A}^{kl,L} = -\,24\pi^2 \int_0^1 dx(1-x)^{2+k} x^{l+1} \rho_{ij}^{V/A,L}(M_\tau^2 x) = -\,4\pi i \oint_{|x|=1} \frac{dx}{x}\, \mathcal{F}_L^{kl}(x)\, \mathcal{D}_{ij}^{V/A,L}(M_\tau^2 x)\,, \tag{3.1}$$

with $\rho_{ij}^{V/A,L}(s) \equiv \mathrm{Im}\,\Pi_{ij}^{V/A,L}(s)/\pi$ being the longitudinal spectral functions. Note that for convenience, we have defined $R_{ij,V/A}^{kl,L}$ without the appropriate CKM and electroweak correction factors. The longitudinal correlators are directly related to corresponding correlators $\Psi_{ij}^{V/A}(s)$ for the divergences of vector and axialvector currents:

$$\Pi_{ij}^{V/A,L}(s) = \frac{1}{s^2}\left[\Psi_{ij}^{V/A}(s) - \Psi_{ij}^{V/A}(0)\right]. \tag{3.2}$$

Furthermore, via the equations of motion for the quark fields, $\Psi_{ij}^{V/A}(s)$ is related to the two-point correlators $\Pi_{ij}^{S/P}(s)$ for the scalar/pseudoscalar currents $J_{ij}^{S/P}(x) = (\bar q_j(\gamma_5)q_i)(x)$, and the constants $\Psi_{ij}^{V/A}(0)$ to the quark condensate from a Ward-identity [35]:

$$\Psi_{ij}^{V/A}(s) = (m_i \mp m_j)^2\, \Pi_{ij}^{S/P}(s)\,, \qquad \Psi_{ij}^{V/A}(0) = -\,(m_i \mp m_j)[\langle \bar q_i q_i\rangle \mp \langle \bar q_j q_j\rangle]\,, \tag{3.3}$$

where m_i and m_j are the masses of the quark flavours q_i and q_j respectively, and $\langle \bar q_i q_i\rangle$ as well as $\langle \bar q_j q_j\rangle$ are the corresponding quark condensates. The upper sign corresponds to the vector/scalar and the lower sign to the axialvector/pseudoscalar case. For simplicity, the vacuum state has been omitted in the condensates, and we would like to point out that the bi-quark operators in eq. (3.3) are *non*-normal-ordered [36, 37]. The theoretical expressions for the longitudinal correlators $\mathcal{D}_{ij}^L(s)$ can either be obtained from refs. [5, 18, 10, 13], or can be calculated straightforwardly from the second of eqs. (2.8) together with eq. (3.2) and the expressions for $\Psi_{ij}^{V/A}(s)$ given in ref. [38]. For further details and the explicit expressions the reader is referred to these references.

Let us now proceed with a numerical investigation of the sum rules of eq. (3.1). In table 2, we display a comparison of theoretical and phenomenological values of the longitudinal contributions $R_{us,A}^{00,L}$, $R_{us,V}^{00,L}$ and $R_{ud,A}^{00,L}$ to the τ sum rule. Due to the global mass squared factor, these results depend sensitively on the quark masses. For definiteness, we have set the strange mass to $m_s(2\,\mathrm{GeV}) = 105\,\mathrm{MeV}$, a value compatible with most recent determinations of m_s both from QCD sum rules [39, 40, 15, 14, 17, 21, 28] and

	$R^{00,L}_{us,A}$	$R^{00,L}_{us,V}$	$R^{00,L}_{ud,A}$
Theory:	-0.144 ± 0.024	-0.028 ± 0.021	$-(7.79 \pm 0.14) \cdot 10^{-3}$
Phenom:	-0.135 ± 0.003	-0.028 ± 0.004	$-(7.77 \pm 0.08) \cdot 10^{-3}$

Table 2: Comparison of theoretical and phenomenological longitudinal contributions to the $(0,0)$ moment of the τ sum rule.

lattice QCD [41, 42, 43, 44]. A detailed comparison to other m_s determinations will be presented in section 6. The light masses m_u and m_d have been fixed from the χPT ratios $m_u/m_d = 0.553 \pm 0.043$ and $m_s/m_d = 18.9 \pm 0.8$ [45]. On the theoretical side, we have also included instanton contributions according to the model used in [21], because they are known to play some rôle in scalar/pseudoscalar finite energy sum rules.

The theoretical uncertainties in table 2 are largely dominated by higher order perturbative corrections. Already for the $(0,0)$ moment, the first and second order corrections are of similar size, while the third order α_s^3 correction is larger than these two. Thus for this moment, in the spirit of asymptotic series, we have cut the perturbative series at the second order. For higher moments, the behaviour is still worse and there the second order turns out to be larger than the first one, which then entails even larger perturbative uncertainties. For this reason, we have not presented detailed results for the higher moments although also in these cases agreement between theoretical and phenomenological results is found within the errors. A detailed discussion of how we estimated the theoretical errors will be given in the next section.

The uncertainties in the phenomenological numbers, on the other hand, are much smaller. For the pseudoscalar spectral functions we have employed the parametrisation by Maltman and Kambor [21]. In the case of the pseudoscalar us channel, the corresponding expression reads

$$s^2 \rho^{A,L}_{us}(s) = 2f_K^2 M_K^4 \delta(s - M_K^2) + 2f_1^2 M_1^4 B_1(s) + 2f_2^2 M_2^4 B_2(s)\,, \tag{3.4}$$

with f_i and M_i being decay constants and masses of the higher exited resonances. $B_i(s)$ is the standard Breit-Wigner resonance shape function

$$B_i(s) = \frac{1}{\pi} \frac{\Gamma_i M_i}{[(s - M_i^2)^2 + \Gamma_i^2 M_i^2]}\,. \tag{3.5}$$

For the pseudoscalar ud channel an equation analogous to (3.4) holds. The masses and widths of the resonances have been taken from the Review of Particle Physics [46] and the decay constants from ref. [21]. For convenience, we have compiled these input parameters in table 3. The dominant uncertainties in the phenomenological results of table 2 are due to the errors in the decay constants f_K and f_π, as well as the variation of the higher resonance decay constants as given in table 3. Changing resonance masses and widths within reasonable ranges only has a minor effect. Therefore, for these quantities, we have not given explicit uncertainties.

The scalar us spectral function has been calculated in ref. [28]. Below 2 GeV the dominant hadronic systems which contribute in this channel are the $K\pi$ and $K\eta'$ states

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	$\pi(1300)$	$\pi(1800)$	$K(1460)$	$K(1800)$
M_i [MeV]	1300	1800	1460	1830
Γ_i [MeV]	400	210	260	250
f_i [MeV]	2.20 ± 0.46	0.19 ± 0.19	21.4 ± 2.8	4.5 ± 4.5

Table 3: Input parameters for the pseudoscalar resonance parametrisation. The masses and widths are taken from [46], whereas the decay constants have been determined in [21].

with $K_0^*(1430)$ being the lowest lying scalar resonance. The scalar us spectral function can then be parametrised in terms of the scalar $F_{K\pi}(s)$ and $F_{K\eta'}(s)$ form factors. These form factors were obtained in [23] from a coupled-channel dispersion-relation analysis. The S-wave $K\pi$ scattering amplitudes which are required in the dispersion relations were available from a description of S-wave $K\pi$ scattering data in the framework of unitarised χPT with resonances [22]. The dominant uncertainty in the scalar us spectral function is due to an integration constant which emerges while solving the coupled channel dispersion relations. In [23] it was decided to fix this constant by demanding $F_{K\pi}(M_K^2 - M_\pi^2) = 1.22 \pm 0.01$, a result which from χPT is known to be very robust, since higher-order chiral corrections are well suppressed [25]. The uncertainty in the phenomenological value for $R_{us,V}^{00,L}$ in table 2 thus corresponds to a variation of this parameter within the given range. Further details on the scalar us spectral function can be found in ref. [28].

Generally, from table 2 one observes that the longitudinal sum rules are very well satisfied. This is partly due to the fact that the subtraction constant $\Psi_{ud}^A(0)$ dominates the ud sum rule and also in the pseudoscalar us sum rule $\Psi_{us}^A(0)$ gives a large contribution. In the leading order in χPT, these subtraction constants are fixed by the pion and kaon pole contributions which led to the famous Gell-Mann-Oakes-Renner (GMOR) relations [47]. Therefore, we have a well known piece which appears on both sides of the sum rule. Nevertheless, the agreement within errors is non-trivial and also points to the fact that the used light quark mass values are reasonable. However, the complete agreement in the scalar us case should be considered as accidental. From table 2 it becomes obvious that the theoretical uncertainties in the flavour-breaking τ sum rule can be greatly improved by replacing the theoretical longitudinal contributions by the corresponding phenomenological ones.

The observed sensitivity of the longitudinal sum rules on the subtraction constants allows to obtain further information on these quantities. For this purpose it is convenient to rewrite the subtraction constants as

$$\Psi_{ud}^A(0) = 2f_\pi^2 M_\pi^2 \, (\, 1 - \delta_\pi \,) \, , \qquad \Psi_{us}^A(0) = 2f_K^2 M_K^2 \, (\, 1 - \delta_K \,) \, , \tag{3.6}$$

where the leading term corresponds to the GMOR relations, and to actually determine the higher order chiral corrections δ_π and δ_K [25, 48]. In addition, from the scalar subtraction constant $\Psi_{us}^V(0)$, the flavour breaking ratio of the quark condensates $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$ can be estimated. Now we have to also vary the strange quark mass, and we chose $m_s(2\,\mathrm{GeV}) = 105 \pm 20\,\mathrm{MeV}$, the central value already advocated above with an uncertainty which is slightly more generous than the ranges obtained in the recent scalar and pseudoscalar sum rule determinations [21, 28].

Again varying all other input parameters according to the ranges presented here and in the next section, we obtain:

$$\delta_\pi = 0.049 \pm 0.030\,, \qquad \delta_K = 0.50 \pm 0.23\,, \qquad \frac{\langle \bar{s}s\rangle}{\langle \bar{u}u\rangle} = 0.80 \pm 0.58\,. \tag{3.7}$$

These results are in very good agreement to the findings of ref. [48], where the values of $\delta_\pi = 0.047 \pm 0.017$ and $\delta_K = 0.61 \pm 0.22$ were estimated in the framework of χPT assuming a given input for the ratio $\langle \bar{s}s\rangle/\langle \bar{u}u\rangle$. However, the latter results of [48] are somewhat more precise, since the uncertainties in eq. (3.7) are again dominated by the large higher order perturbative corrections and now in addition by the error on the strange quark mass. Besides from the scalar subtraction constant, the ratio $\langle \bar{s}s\rangle/\langle \bar{u}u\rangle$ can also be directly inferred from δ_K, and from δ_π by inverting the line of reasoning applied in ref. [48]. The central values found in this way are $\langle \bar{s}s\rangle/\langle \bar{u}u\rangle = 0.63$ and 0.75 respectively, however, with uncertainties which are even larger than the one presented in eq. (3.7). Nevertheless, it is gratifying to observe that the central results for $\langle \bar{s}s\rangle/\langle \bar{u}u\rangle$ display good consistency.

4. Strange quark mass

Our determination of the strange quark mass from the SU(3)-breaking differences of (2.10) proceeds in complete analogy to the previous analyses [13, 17], with the exception that now we have replaced the longitudinal contributions to the sum rule by phenomenological parametrisations as discussed above. Solving for the strange mass, eq. (2.10) can be written as

$$m_s^2(M_\tau) = \frac{M_\tau^2}{18(1-\epsilon_d^2)\Delta_{kl}^{L+T}(a_\tau)} \left\{ \frac{\delta R_\tau^{kl}}{S_{\rm EW}} - \delta R_{\tau,\rm phen}^{kl,L} - \delta R_{\tau,D\geq 4}^{kl,L+T} \right\}, \tag{4.1}$$

where $\epsilon_d \equiv m_d/m_s$. The phenomenological contribution containing the longitudinal spectral functions is given by

$$\delta R_{\tau,\rm phen}^{kl,L} = R_{ud,V}^{kl,L} + R_{ud,A}^{kl,L} - R_{us,V}^{kl,L} - R_{us,A}^{kl,L}\,, \tag{4.2}$$

with $R_{ij,V/A}^{kl,L}$ as defined in eq. (3.1) and calculated using the scalar and pseudoscalar spectral functions discussed in the last section.

The perturbative QCD correction $\Delta_{kl}^{L+T}(a_\tau)$, associated with the dimension-2 contribution, depends only on $a_\tau \equiv \alpha_s(M_\tau)/\pi$ and has been discussed in great detail in refs. [10, 13]. Since we have subtracted the badly behaved longitudinal contribution, the remaining part displays a better convergence at higher orders. In contour-improved perturbation theory [49], and taking as a central value $\alpha_s(M_\tau) = 0.334$ [8], the expansion for the relevant moments takes the form

$$\begin{aligned}
\Delta_{00}^{L+T}(a_\tau) &= 0.753 + 0.214 + 0.065 - 0.051 + \cdots = 0.981\,, \\
\Delta_{10}^{L+T}(a_\tau) &= 0.912 + 0.334 + 0.192 + 0.056 + \cdots = 1.493\,, \\
\Delta_{20}^{L+T}(a_\tau) &= 1.055 + 0.451 + 0.330 + 0.189 + \cdots = 2.024\,, \\
\Delta_{30}^{L+T}(a_\tau) &= 1.190 + 0.571 + 0.484 + 0.352 + \cdots = 2.597\,, \\
\Delta_{40}^{L+T}(a_\tau) &= 1.324 + 0.697 + 0.657 + 0.551 + \cdots = 3.228\,.
\end{aligned} \tag{4.3}$$

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Parameter	Value	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$
$m_s(M_\tau)$		197.3	164.0	136.6	115.2	98.6
δR_τ^{kl}	[17]	$^{+52.7}_{-67.7}$	$^{+22.2}_{-25.4}$	$^{+13.1}_{-14.5}$	$^{+9.3}_{-10.2}$	$^{+7.3}_{-7.9}$
$\lvert V_{us}\rvert$	0.2225 ± 0.0021	$^{+28.8}_{-32.6}$	$^{+14.6}_{-16.0}$	$^{+9.4}_{-10.1}$	$^{+6.7}_{-7.1}$	$^{+5.3}_{-5.6}$
$\mathcal{O}(\alpha_s^3)$	$2\times\mathcal{O}(\alpha_s^3)$ no $\mathcal{O}(\alpha_s^3)$	$^{+5.9}_{-5.4}$	$^{-3.1}_{+3.3}$	$^{-6.0}_{+6.9}$	$^{-7.1}_{+8.6}$	$^{-7.5}_{+9.6}$
ξ	1.5 0.75	$^{-11.7}_{+29.0}$	$^{-1.5}_{+11.8}$	$^{+3.5}_{+4.2}$	$^{+6.3}_{-0.1}$	$^{+7.9}_{-2.7}$
$\alpha_s(M_\tau)$	0.334 ± 0.022	$^{+14.2}_{-9.8}$	$^{+5.0}_{-3.0}$	$^{+1.1}_{+0.1}$	$^{-0.9}_{+1.8}$	$^{-2.0}_{+2.8}$
$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$	0.8 ± 0.2	$^{-0.8}_{+0.8}$	$^{-3.6}_{+3.5}$	$^{-5.6}_{+5.3}$	$^{-7.2}_{+6.8}$	$^{-8.4}_{+7.9}$
$\rho_{us}^{\text{scalar}}$	see text	$^{-2.0}_{+1.8}$	$^{-0.8}_{+0.7}$	$^{-0.4}_{+0.3}$	$^{-0.2}_{+0.2}$	$^{-0.1}_{+0.1}$
f_K	$113 \pm 1\,\text{MeV}$	$^{-1.1}_{+1.1}$	$^{-0.8}_{+0.8}$	$^{-0.6}_{+0.6}$	$^{-0.5}_{+0.5}$	$^{-0.5}_{+0.4}$
$f_{K(1460)}$	$21.4 \pm 2.8\,\text{MeV}$	$^{-1.1}_{+0.9}$	$^{-0.4}_{+0.4}$	$^{-0.2}_{+0.2}$	$^{-0.1}_{+0.1}$	$^{-0.1}_{+0.1}$
Total		$^{+68.4}_{-76.9}$	$^{+29.9}_{-30.6}$	$^{+18.9}_{-19.5}$	$^{+17.2}_{-16.0}$	$^{+17.5}_{-15.2}$

Table 4: Central results for $m_s(M_\tau)$ corresponding to the unitarity fit for $|V_{us}|$ extracted from the different moments, as well as ranges for the main input parameters and resulting uncertainties for m_s. For a detailed description see the discussion in the text.

The last term in the series, corresponding to the order α_s^3 correction, has not yet been calculated. In eq. (4.3), we have employed an estimate based on assuming a geometrical growth of the Adler function $\mathcal{D}^{L+T}(s)$ [13]. Finally, $\delta R_{\tau,D\geq 4}^{kl,L+T}$ are higher-dimensional operator corrections in the framework of the operator product expansion, the dominant one being the $D = 4$ correction resulting from the quark condensate. Explicit expressions for these contributions can be found in ref. [13]. In our numerical analysis below, the $D = 4$ light up and down quark mass corrections as well as the $D = 6$ condensates have also been included, although their influence on the value of m_s is insignificant.

Due to cancellations in the difference (2.10), it was already found in the earlier analyses [13, 17, 21] that the strange mass resulting from eq. (4.1) depends sensitively on the value for $|V_{us}|$. Therefore, in what follows we shall perform two separate extractions of m_s. In the first one, like in ref. [17], the value for $|V_{us}|$ is taken from the Particle Data Group unitarity fit of the CKM matrix [46] yielding $|V_{us}| = 0.2225 \pm 0.0021$. In table 4, we show a detailed account of our results. The first row displays the values of $m_s(M_\tau)$ obtained from the different moment sum rules and central values for all input parameters. In the following rows, we have listed those input parameters which dominantly contribute to the uncertainty on m_s, the ranges for these parameters used in our analysis and the resulting shift in m_s. Only those parameters have been included in the list which at least for one moment yield a shift of m_s larger than 1 MeV. Finally, in the last row, we display the total error that results from adding the individual uncertainties in quadrature.

Let us now discuss the different uncertainties in more detail. The experimental results for δR_τ^{k0} with $k = 0, \ldots, 4$, as well as the errors corresponding to a variation of $|V_{us}|$, have been presented in ref. [17] and are taken over from there. They dominate for low k and become smaller if k is increased. For the uncertainties related to higher order perturbative corrections we have considered three different sources. As already mentioned above, the order α_s^3 correction to $\Delta_{kl}^{L+T}(a_\tau)$ has not yet been calculated and thus we rely on estimates of this correction [5, 49]. To be conservative, for our error estimate we have chosen two extreme cases, one where this correction is doubled and the other where it is removed completely. The corresponding variations of m_s are given in table 4. The second source is the renormalisation scale dependence which arises due to the missing higher orders which would make physical quantities independent of the scale. Like in ref. [49], we have parametrised this dependence by the parameter $\xi \equiv \mu/\sqrt{-s}$, and we have varied this parameter in the range $0.75 < \xi < 1.5$. The upper range is slightly reduced compared to the previous analyses, but since we have included explicitly the variation of the $\mathcal{O}(\alpha_s^3)$ correction, we consider this reduction justified. Finally, our input value for $\alpha_s(M_\tau) = 0.334 \pm 0.022$ [17] reflects itself in an additional uncertainty for the m_s determination.

The only parameter which matters for the uncertainty related with the $D = 4$ contribution in the OPE is the flavour-breaking ratio of the quark condensates $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$. For this parameter we have chosen the range $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.8 \pm 0.2$ which includes most values found in the literature [50, 51, 52, 38, 53]. Certainly, this value is also compatible with the results obtained in section 3. For increasing k, higher order terms in the OPE become more important, and thus for the larger k values this error, together with the perturbative QCD uncertainty, dominates. Although we have included the $D = 6$ contribution according to the expressions given in [13] in our numerical analysis, for the resulting value of m_s and thus also for the uncertainty they can be neglected.

The final uncertainties included in table 4 concern the variation of the phenomenological scalar and pseudoscalar spectral functions. The variations of the corresponding parameters have already been discussed in section 3, where we have analysed τ-like sum rules for the longitudinal contributions separately. The dominant sources of error are the scalar us spectral function as discussed briefly in section 3, and in more detail in ref. [28], as well as the decay constants in the pseudoscalar us channel f_K and $f_K(1460)$. However, as can be seen from table 4, the influence on the error of the strange quark mass is very minor. In addition, it again corroborates the reduction of the final uncertainty of m_s achieved through the replacement of theoretical by phenomenological scalar and pseudoscalar longitudinal contributions. Variations of all remaining input parameters lead to changes in m_s of less than 1 MeV, and have thus been neglected.

To calculate a final result for $m_s(M_\tau)$, we have used a weighted average over the different moments where as the individual error for each moment, we have taken the larger one. This prescription then yields:

$$m_s(M_\tau) = 121.7 \pm 17.2\,\mathrm{MeV} \quad \Rightarrow \quad m_s(2\,\mathrm{GeV}) = 117 \pm 17\,\mathrm{MeV}\,. \tag{4.4}$$

The result on the right-hand side is the value of m_s evolved to a renormalisation scale of 2 GeV. As the uncertainty we have taken the smallest error of one individual moment, (3,0) in this case. Taking the correlation of the different moments into account, the final uncertainty could still be slightly reduced. However, since the moments are strongly correlated, this would give little improvement, and thus at present we have not incorporated this approach.

Parameter	Value	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$
$m_s(M_\tau)$		150.8	141.5	122.5	105.3	90.8
δR_τ^{kl}	experiment	$^{+61.9}_{-103.3}$	$^{+25.0}_{-30.1}$	$^{+14.5}_{-16.4}$	$^{+10.2}_{-11.3}$	$^{+7.9}_{-8.7}$
$\lvert V_{us}\rvert$	0.2196 ± 0.0026	$^{+42.1}_{-56.6}$	$^{+20.3}_{-23.5}$	$^{+12.7}_{-14.2}$	$^{+8.9}_{-9.8}$	$^{+7.0}_{-7.6}$
$\mathcal{O}(\alpha_s^3)$	$2\times\mathcal{O}(\alpha_s^3)$ no $\mathcal{O}(\alpha_s^3)$	$^{+4.3}_{-3.9}$	$^{-2.6}_{+2.8}$	$^{-5.3}_{+6.1}$	$^{-6.5}_{+7.9}$	$^{-6.9}_{+8.9}$
ξ	1.5 0.75	$^{-8.1}_{+20.2}$	$^{-1.1}_{+9.7}$	$^{+3.2}_{+3.6}$	$^{+5.8}_{-0.1}$	$^{+7.3}_{-2.5}$
$\alpha_s(M_\tau)$	0.334 ± 0.022	$^{+10.1}_{-7.0}$	$^{+4.2}_{-2.5}$	$^{+1.0}_{+0.1}$	$^{-0.8}_{+1.6}$	$^{-1.9}_{+2.5}$
$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$	0.8 ± 0.2	$^{-1.2}_{+1.1}$	$^{-4.3}_{+4.1}$	$^{-6.3}_{+5.9}$	$^{-8.0}_{+7.4}$	$^{-9.2}_{+8.5}$
$\rho_{us}^{\mathrm{scalar}}$	see text	$^{-2.5}_{+2.2}$	$^{-0.9}_{+0.8}$	$^{-0.4}_{+0.4}$	$^{-0.2}_{+0.2}$	$^{-0.2}_{+0.1}$
f_K	$113 \pm 1\,\mathrm{MeV}$	$^{-1.4}_{+1.4}$	$^{-0.9}_{+0.9}$	$^{-0.7}_{+0.7}$	$^{-0.6}_{+0.6}$	$^{-0.5}_{+0.5}$
$f_{K(1460)}$	$21.4 \pm 2.8\,\mathrm{MeV}$	$^{-1.3}_{+1.2}$	$^{-0.5}_{+0.4}$	$^{-0.2}_{+0.2}$	$^{-0.1}_{+0.1}$	$^{-0.1}_{+0.1}$
Total		$^{+78.4}_{-118.4}$	$^{+34.2}_{-38.6}$	$^{+21.4}_{-23.2}$	$^{+18.3}_{-18.1}$	$^{+17.9}_{-16.6}$

Table 5: Central results for $m_s(M_\tau)$ corresponding to the PDG average for $|V_{us}|$ extracted from the different moments, as well as ranges for the main input parameters and resulting uncertainties for m_s. For a detailed description see the discussion in the text.

Comparing our central strange mass average to the m_s values obtained for the individual moments, it is evident that there is a strong dependence of m_s on the number of the moment k. For our approach, this dependence is even stronger than in the analysis of ref. [17]. The reason lies in the fact that the scalar and pseudoscalar sum rules by themselves are satisfied for smaller strange masses than the full τ sum rule. Thus, when replacing the theoretical scalar and pseudoscalar spectral functions by phenomenological ones, the resulting m_s moves up. Nevertheless, one should clearly state that the experimental uncertainties, especially for the low moments are still large, and the difference between the individual moment results and the average strange mass for all values lies in the range of one standard deviation.

As was already mentioned above, the central PDG average for $|V_{us}| = 0.2196 \pm 0.0026$ [46], which is based on the analyses [54, 55, 56], lies more than one sigma below the value extracted by requiring unitarity for the first row of the CKM matrix. Due to this large difference, we consider it justified to repeat our strange mass determination for this value of $|V_{us}|$ separately. To properly perform the respective analysis, one would in principle need the experimental results for the R-ratios $R_{\tau,V+A}^{kl}$ and $R_{\tau,S}^{kl}$ for all moments separately. However, as yet, these are only published for the lowest $(0,0)$ moment, whereas for the higher moments the combined result δR_τ^{kl} based on the unitarity value for $|V_{us}|$ is solely available [17]. Since in ref. [17] the uncertainty from varying $|V_{us}| = 0.2225 \pm 0.0021$ was listed separately, we can nevertheless work around this problem by rescaling δR_τ^{kl} appropriately to the above PDG average.

The result of our second analysis is displayed in table 5. Like in table 4, again the first row contains the individual moment results for central input parameters, then the variations of m_s while varying the dominant input parameters are tabulated, and the last row gives the combined uncertainty by adding all errors in quadrature. The variation of the various input parameters proceeds in complete analogy to the unitarity case discussed above. Again performing a weighted average of the individual moment results treating the uncertainties like for the unitarity $|V_{us}|$, the resulting strange quark mass turns out to be

$$m_s(M_\tau) = 107.0 \pm 17.9\,\mathrm{MeV} \quad \Rightarrow \quad m_s(2\,\mathrm{GeV}) = 103 \pm 17\,\mathrm{MeV}\,. \tag{4.5}$$

We observe that for the lower value of $|V_{us}|$, we also find m_s to be lowered by roughly one sigma, but nevertheless certainly compatible with the result of eq. (4.4). As table 5 shows, here the k dependence of m_s is weaker than for the unitarity case, the reason being that now the central m_s lies much closer to the value preferred by the scalar and pseudoscalar sum rules. Then the replacement of theoretical by phenomenological spectral functions only shifts the resulting m_s by a smaller amount.

5. A novel route to $|V_{us}|$

Since the quantities δR_τ^{kl} of eq. (2.10) are strongly dependent on $|V_{us}|$, especially for low k, now we want to turn the table and investigate the potential of determining $|V_{us}|$ from the SU(3)-breaking τ sum rule. For this novel approach, we require a value for the strange mass from other sources as an input so that we are in a position to calculate δR_τ^{kl} from theory. Like in section 3, we have used $m_s(2\,\mathrm{GeV}) = 105 \pm 20\,\mathrm{MeV}$ which is compatible with the recent sum rule determinations [39, 40, 21, 28] and also with the lattice results [41, 42, 43, 44].[1]

Our results for the theoretical prediction of δR_τ^{kl} are presented in table 6. Like our results for the strange mass, the first row shows our central values for δR_τ^{kl} and below the uncertainties resulting from varying the various input parameters within their ranges, now including the strange mass, are listed. Again, the last row gives the total uncertainty by adding all errors in quadrature. One observes that the errors on δR_τ^{kl} for all moments are dominated by the present uncertainty in the strange quark mass, but also the uncertainties resulting from higher order QCD corrections play some rôle.

Parameter	Value	(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
$\delta R_{\tau,th}^{kl}$	central	0.229	0.270	0.320	0.375	0.436
$m_s(2\,\mathrm{GeV})$	$105 \pm 20\,\mathrm{MeV}$	$^{+0.028}_{-0.023}$	$^{+0.043}_{-0.036}$	$^{+0.058}_{-0.048}$	$^{+0.075}_{-0.062}$	$^{+0.094}_{-0.077}$
$\mathcal{O}(\alpha_s^3)$	$2\times\mathcal{O}(\alpha_s^3)$ / no $\mathcal{O}(\alpha_s^3)$	$^{-0.004}_{+0.004}$	$^{+0.004}_{-0.004}$	$^{+0.013}_{-0.013}$	$^{+0.024}_{-0.024}$	$^{+0.038}_{-0.038}$
ξ	1.5 / 0.75	$^{+0.007}_{-0.013}$	$^{+0.001}_{-0.012}$	$^{-0.007}_{-0.007}$	$^{-0.018}_{+0.000}$	$^{-0.032}_{+0.013}$
$\alpha_s(M_\tau)$	0.334 ± 0.022	$^{-0.007}_{+0.006}$	$^{-0.005}_{+0.003}$	$^{-0.001}_{-0.001}$	$^{+0.004}_{-0.007}$	$^{+0.011}_{-0.014}$
$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$	0.8 ± 0.2	$^{+0.002}_{-0.002}$	$^{+0.010}_{-0.010}$	$^{+0.018}_{-0.018}$	$^{+0.024}_{-0.024}$	$^{+0.030}_{-0.031}$
$\rho_{us}^{\mathrm{scalar}}$	see text	$^{+0.004}_{-0.004}$	$^{+0.002}_{-0.002}$	$^{+0.001}_{-0.001}$	$^{+0.001}_{-0.001}$	$^{+0.001}_{-0.001}$
f_K	$113 \pm 1\,\mathrm{MeV}$	$^{+0.002}_{-0.002}$	$^{+0.002}_{-0.002}$	$^{+0.002}_{-0.002}$	$^{+0.002}_{-0.002}$	$^{+0.002}_{-0.002}$
$f_{K(1460)}$	$21.4 \pm 2.8\,\mathrm{MeV}$	$^{+0.002}_{-0.002}$	$^{+0.001}_{-0.001}$	$^{+0.001}_{-0.001}$	$^{+0.000}_{-0.000}$	$^{+0.000}_{-0.000}$
Total	error	$^{+0.030}_{-0.028}$	$^{+0.044}_{-0.040}$	$^{+0.062}_{-0.054}$	$^{+0.083}_{-0.074}$	$^{+0.107}_{-0.098}$

Table 6: Central results for R_τ^{kl} extracted from theory, as well as ranges for the main input parameters and corresponding resulting uncertainties.

[1] Further references to original works can be found in these reviews.

Since experimental results for the individual R-ratios $R_\tau = 3.642 \pm 0.012$ and $R_{\tau,S} = 0.1625 \pm 0.0066$ [57] are available, we can employ our result for the $(0,0)$ moment

$$\delta R_{\tau,th} = 0.229 \pm 0.030\,, \tag{5.1}$$

to calculate the value of $|V_{us}|$. Assuming unitarity of the CKM matrix in order to express the CKM matrix element $|V_{ud}|$ in terms of $|V_{us}|$, we find

$$|V_{us}| = 0.2179 \pm 0.0044_{\rm exp} \pm 0.0009_{\rm th} = 0.2179 \pm 0.0045\,. \tag{5.2}$$

The first given error is the experimental uncertainty due to R_τ and $R_{\tau,S}$ whereas the second error stems from the theoretical value for δR_τ. Our result for $|V_{us}|$ is in good agreement with the PDG average although at present the uncertainty is roughly twice as large and it is only one sigma away from $|V_{us}|$ as obtained from the CKM unitarity fit. However, the current uncertainty is dominated by the experimental uncertainty in $R_{\tau,S}$, and if this shrinks by a factor of two, also a corresponding reduction in the uncertainty for $|V_{us}|$ would be reached. Then the determination of $|V_{us}|$ from the τ sum rule could reach a precision better than the current world average. In addition, if the value of m_s will become better known in the future, there is also room for improvement on the theory side.

A second possibility to calculate $|V_{us}|$ from τ decays into strange particles with net strangeness would be to directly employ the theoretical prediction for $R_{\tau,S}/|V_{us}|^2$. Estimating the uncertainties in analogy to table 6, we obtain

$$\frac{R_{\tau,S}}{|V_{us}|^2} = 3.395 \pm 0.069\,. \tag{5.3}$$

Since for $R_{\tau,S}$, contrary to δR_τ, also the $D = 0$ contribution $\delta^{(0)}$ is present, in this case the uncertainty is dominated by the perturbative QCD error, although also the error on the strange quark mass matters somewhat. Together with the experimental value $R_{\tau,S} = 0.1625 \pm 0.0066$, again we are in a position to extract $|V_{us}|$ with the result:

$$|V_{us}| = 0.2188 \pm 0.0044_{\rm exp} \pm 0.0022_{\rm th} = 0.2188 \pm 0.0049\,. \tag{5.4}$$

We observe that our findings of eqs. (5.2) and (5.4) are completely compatible. The experimental uncertainty in the latter result is identical to the one of (5.2), whereas the theory error, due to the contribution from perturbation theory, is larger. To conclude, when improved experimental data will be available, we expect that with our first approach a precise determination of $|V_{us}|$ from δR_τ can be achieved.

6. Comparison with other analyses

Our determination of the strange quark mass from hadronic τ decays presented above can readily be compared to the recent analysis [17], the most notable difference being our replacement of the badly behaved longitudinal contributions by phenomenological parametrisations. In ref. [17] the final result for the strange mass was $m_s(M_\tau) = 120^{+21}_{-26}$ assuming the value for $|V_{us}|$ as obtained from a fit to the unitarity of the CKM matrix. This finding is in perfect agreement to our result (4.4) in this case, although through the usage of the phenomenological scalar and pseudoscalar spectral functions, a reduction of the uncertainties could be achieved in our investigation.

Other analyses of m_s from hadronic τ decays, which have been published in the last years, can be found in refs. [11, 12, 13, 14, 15, 16]. A detailed comparison of these analyses has recently been worked out by Maltman in ref. [58]. There it was observed that, once all analyses are updated by using common values for $R_{\tau,S}$ and of the CKM matrix element $|V_{us}|$, the resulting strange mass values all cluster around $m_s(2\,\mathrm{GeV}) = 115\,\mathrm{MeV}$ for a unitarity $|V_{us}|$ and $m_s(2\,\mathrm{GeV}) = 102\,\mathrm{MeV}$ for the PDG average, being in perfect agreement to our findings of eqs. (4.4) and (4.5) respectively.

Sum rule determinations of the strange quark mass have also been performed on the basis of the divergence of the vector or axialvector spectral functions alone [59, 60, 21, 28]. Being directly proportional to m_s^2, these sum rules are very sensitive to the strange mass and the relevant spectral functions are identical to the ones employed to replace the scalar and pseudoscalar longitudinal contribution to the τ sum rule. The outcomes of these analyses were $m_s(2\,\mathrm{GeV}) = 100 \pm 12\,\mathrm{MeV}$ for the pseudoscalar finite energy sum rule [21], and $m_s(2\,\mathrm{GeV}) = 99 \pm 16\,\mathrm{MeV}$ in the case of the scalar Borel sum rule [28]. Again, these results are in very good agreement to our value for m_s of eq. (4.5) for the PDG average for $|V_{us}|$, whereas in the case of the unitarity $|V_{us}|$ the result from the τ sum rule is roughly one sigma higher. However, at present, due to the large uncertainties no further conclusions from this slight discrepancy can be reached.

The status of the extraction of m_s from the hadronic e^+e^- cross section is less clear. In an analysis which took into account flavour-breaking differences, analogous to the τ sum rule, Narison [61] obtained $m_s(2\,\mathrm{GeV}) = 144 \pm 21$ MeV. In refs. [39, 62] it was then pointed out that large isospin breaking corrections significantly lower the result for the strange mass to about $m_s(2\,\mathrm{GeV}) = 95$ MeV and yield considerably larger uncertainties of the order of 45 MeV. A recent improved analysis taking into account further SU(3)-breaking differences of hadronic correlators now yields a value of $m_s(2\,\mathrm{GeV}) = 129 \pm 24$ MeV [40]. Nevertheless, in our opinion further work in this channel is needed, before a definite conclusion can be reached [63].

Recent reviews of determinations of the strange quark mass from lattice QCD have been presented in refs. [41, 42, 43, 44], with the conclusions $m_s(2\,\mathrm{GeV}) = 108 \pm 15\,\mathrm{MeV}$ and $m_s(2\,\mathrm{GeV}) = 90 \pm 20$ MeV in the quenched and unquenched cases respectively. In the unquenched case the error in these results is dominated by the uncertainty resulting from dynamical fermions, whereas the calculations of m_s in the quenched theory, based for example on the kaon mass, are already very precise. However, also in the quenched lattice simulations additional uncertainties arise due to the fact that strange masses extracted from different quantities display systematic deviations [44]. Nevertheless, the agreement between lattice QCD and QCD sum rule determinations of m_s is already rather satisfactory.

In figure 1, we display a summary of the status of strange mass determinations from QCD sum rules and lattice QCD. The two open circles represent the results of this work given in eqs. (4.4) and (4.5). The full circles correspond to other sum rule determinations discussed in this section, while the full squares are averages for lattice QCD simulations in

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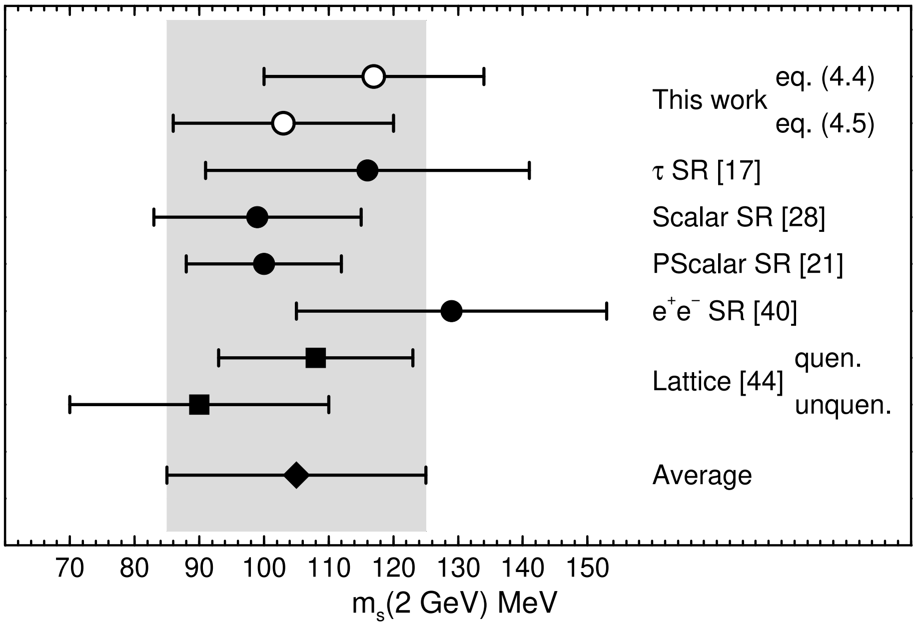

Figure 1: Summary of results for $m_s(2\,\mathrm{GeV})$.

the quenched and unquenched case. Finally, the full diamond corresponds to our average value $m_s(2\,\mathrm{GeV}) = 105 \pm 20\,\mathrm{MeV}$ which we have used in sections 3 and 5 as an input.

7. Conclusion

The most severe drawback of determinations of the strange quark mass from the flavour SU(3)-breaking difference (2.10) so far has been the bad behaviour of the longitudinal contributions (3.1) within perturbative QCD. This drawback could be circumvented by replacing the badly behaved longitudinal contributions with corresponding phenomenological parametrisations of the relevant spectral functions. To gain confidence in this treatment, in section 3 we verified that the scalar and pseudoscalar τ-like finite energy sum rules are satisfied by themselves within the uncertainties, which however are large for the theoretical expressions.

Already in previous analyses it was found that the extracted strange mass depends sensitively on the value of the CKM matrix element $|V_{us}|$ which is used in eq. (2.10). Thus, in our numerical analysis of section 4, we decided to consider two different cases: in the first one, like in ref. [17], we employed the value $|V_{us}| = 0.2225 \pm 0.0021$ which results from a fit to the unitarity of the CKM matrix [46]. At a scale of 2 GeV our resulting strange mass (4.4) was found to be

$$m_s(2\,\mathrm{GeV}) = 117 \pm 17\,\mathrm{MeV}\,. \tag{7.1}$$

On the other hand using the PDG central average $|V_{us}| = 0.2196 \pm 0.0026$, which is based on the analyses [54, 55, 56], the strange mass obtained is lowered by roughly one standard deviation and takes the value

$$m_s(2\,\mathrm{GeV}) = 103 \pm 17\,\mathrm{MeV}\,. \tag{7.2}$$

One should, however, remark that these results are obtained from an average over the different $(k, 0)$ moments which have been analysed and that the individual results for the strange mass display a sizeable dependence on the number of the moment. Nevertheless, the deviations to our central averages are within one sigma for all individual moments.

Taking advantage of the strong sensitivity of the flavour-breaking τ sum rule on $|V_{us}|$, in section 5 we inverted the line of reasoning and determined the CKM matrix element $|V_{us}|$ from the same sum rule by assuming an average for the strange quark mass as extracted from other sources. The result thus obtained is

$$|V_{us}| = 0.2179 \pm 0.0045 \,, \tag{7.3}$$

where the error is largely dominated by the experimental result $R_{\tau,S} = 0.1625 \pm 0.0066$ [57]. A reduction of this uncertainty by a factor of two would therefore result in a corresponding reduction in the error on $|V_{us}|$, which would lead to a determination more precise than the current PDG average. Furthermore, precise experimental measurements of $R^{kl}_{\tau,S}$ and the SU(3)-breaking differences δR^{kl}_τ would open the possibility to determine both m_s and $|V_{us}|$ simultaneously. This can hopefully be achieved with the BABAR and BELLE τ data samples in the near future.

Such an independent access on $|V_{us}|$ would be extremely welcome in view of the fact that at present the situation about unitarity in the first row of the quark-mixing matrix is slightly confusing. Whereas with the PDG average $|V_{us}| = 0.2196 \pm 0.0026$, based solely on the analysis of K_{e3} decays, unitarity would be violated at the two sigma level, preliminary results of the E865 experiment at Brookhaven National Laboratory [64] would correspond to $|V_{us}| = 0.2278 \pm 0.0029$ [65], thus being completely compatible with the unitarity of the CKM matrix.[2] This preliminary value would, however, be about two standard deviations away from our result for $|V_{us}|$ presented above. In addition, one should keep in mind that also shifts in the value for $|V_{ud}|$ are certainly possible. Anyhow, with the upcoming improvements on this subject, we are confident that the question about unitarity in the CKM matrix can be resolved in the near future.

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